The Lagrange Multiplier Method

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The Lagrange multiplier method is a strategy for solving constrained optimizations named after the mathematician Joseph-Louis Lagrange.

It consists of transforming a constrained optimization into an unconstrained optimization by incorporating each constraint through a unique associated Lagrange multiplier.

Imagine we want to solve the following constrained optimization:

Maximize $f(x, y) = xy$ such that $x + y = 100$

The first step we take is to set the Lagrangian function, which can be written as[1]:

$$\mathcal{L}(x, y, \lambda) = xy + \lambda[100–(x + y)] \tag{1}$$

We call λ the Lagrange multiplier. [2]

The second step is to optimize the Lagrangian function as an unconstrained optimization problem. That means that we are looking for (x, y, λ) that make the first derivatives of L equal to zero:

$$d\mathcal{L}/dx = y - \lambda = 0$$

$$d\mathcal{L}/dy = x - \lambda = 0$$

$$d\mathcal{L}/d\lambda = 100 - (x + y) = 0$$

Here we have a system of three equations with three unknowns. Notice that the third equation represents the constraint.

The last step is to solve the system and get the values of x, y and λ.

Check that here $x^* = y^* = \lambda^* = 50$.

[1] Or we could also write it like $\mathcal{L}(x, y, \lambda) = xy - \lambda[x + y - 100]$. Notice that the sign of λ and 100 are the same inside each expression.

[2] If we had more than one constrained, we will need more than one multiplier.

In economics we call λ the shadow price of the constraint. It tells us how much the function we want to optimize, $f(.)$, changes when we relax the constraint, 100. This provides meaningful information for economic problems as you will see in the following examples.

Example: Producer's cost minimization

Imagine we want to minimize the cost of producing 121 cars with a production function $q = LK$. The wage rate is \$5 and the rental rate is \$20.

The constrained optimization then is: minimize $C = 5L + 20K$ such that $q = LK = 121$.

The first step is to set the Lagrangian function as[3]:

$$\mathcal{L}(L, K, \lambda) = 5L + 20K + \lambda[121 - LK]$$

The second step is to take derivatives and set them equal to zero:

$$d\mathcal{L}/dL = 5 - \lambda K = 0$$

$$d\mathcal{L}/dK = 20 - \lambda L = 0$$

$$d\mathcal{L}/d\lambda = 121 - LK = 0$$

Here we have a system of three equations and three unknowns. After dividing the first equation by the second equation we get:

$K/L = 5/20$

This is the $MRTS = w/r$ optimality condition.

From here we know $K = 0.25L$. Plugging it into the third equation we get $L^* = 22, K^* = 5.5$ and $\lambda^* = 0.9091$

If you increase the quantity of cars you produce from 121 to 122, this will require you to change how much L and K you use and find the new cost-minimizing combination for this higher quantity. Assuming you do that, what is the cost of producing one more car? In other words, what is the marginal cost? The shadow price tells us that. In this model, the shadow price is the marginal cost.

[3] Check that the Lagrangian function here is a function of L, K and λ

Example 2: Consumer Optimization

Imagine we want to find the optimal bundle of a consumer whose preferences are represented by $U(x,y) = x^{1/2}y^{1/2}$ given that the prices are $p_x = 2$, $p_y = 2$ and their income is $M = 10$. We have here a constrained optimization, since we want to maximize the utility function given the budget constrain.

First, we write the budget constraint:

$p_x x + p_y y = M$ as $2x + 2y = 10$ once we plug the prices and the income into the budget line equation.

Now, we can write the Lagrangian function:

$$\mathcal{L}(x, y, \lambda) = x^{1/2}y^{1/2} + \lambda[10 - (2x + 2y)]$$

Here we are ready to take the first derivatives and set them equal to zero:

$$d\mathcal{L}/dx = x^{-1/2}y^{1/2}/2 - 2\lambda = 0 \text{ as } x^{-1/2}y^{1/2}/2 = 2\lambda$$

$$d\mathcal{L}/dy = x^{1/2}y^{-1/2}/2 - 2\lambda = 0 \text{ as } x^{1/2}y^{-1/2}/2 = 2\lambda$$

$$d\mathcal{L}/d\lambda = 2x + 2y - 10 = 0$$

To solve the system, we divide the first equation by the second equation:

$$\frac{x^{-1/2}y^{1/2}/2}{x^{1/2}y^{-1/2}/2} = 2/2$$

Notice that the Right-Hand Side (RHS) is the $MRS = MU_x/MU_y$ and the Left-Hand Side (LHS) is the $MktRS = p_x/p_y$, so this expression is the optimality condition in a consumer problem, $MRS = MktRS$.

After simplifying, we get $y/x = 1$ or $y = x$. We now plug this expression on our Budget line and solve for x and y. The optimal bundle is $x^* = 2.5$ and $y^* = 2.5$.

Plugging those values in any of the first equations we can solve for λ, $\lambda^* = 0.25$

The shadow price of Income (the marginal utility of one dollar) is 0.25.

General Form

Imagine we want to solve the following constrained optimization:

Max $f(x, y)$ such that $g(x, y) = B$

Here we want to find the pair (x, y) that maximizes the function $f(.)$ such that it is also true that after applying the function $g(.)$ to the same pair (x, y) we get B.

We set the Lagrangian function as[4]:

$$\mathcal{L} = f(x, y) + \lambda[B - g(x, y)]$$

Next we take derivatives:

$$d\mathcal{L}/dx = \partial f(x, y)/\partial x - \lambda \partial g(x, y)/\partial x = 0$$

$$d\mathcal{L}/dy = \partial f(x, y)/\partial y - \lambda \partial g(x, y)/\partial y = 0$$

$$d\mathcal{L}/d\lambda = g(x, y) - B = 0$$

We get a system of three equations and three unknowns (x, y, λ)

If we isolate λ from the first and second equation we get[5] :

$$\lambda^* = \frac{\partial f/\partial x(x^*, y^*)}{\partial g/\partial x(x^*, y^*)} = \frac{\partial f/\partial y(x^*, y^*)}{\partial g/\partial y(x^*, y^*)}$$

The shadow price on B, λ, then reflects how much higher $f(.)$ would go if we were to increase B in a marginal unit relative to how much $g(.)$ has changed.

[4] We can also write it as $\mathcal{L} = f(x, y) - \lambda[g(x, y) - B]$

[5] This reads as: the optimal lambda (λ) is equal to the change in $f(.)$ that occurs due to a marginal change in x evaluated at the optimal (x^*, y^*) divided by the change in $g(.)$ that occurs due to a marginal change in x evaluated at the optimal (x^*, y^*) from the first equation.

From the second equation the optimal lambda (λ) is equal to the change in $f(.)$ that occurs due to a marginal change in y evaluated at the optimal (x^*, y^*) divided by the change in $g(.)$ that occurs due to a marginal change in y evaluated at the optimal (x^*, y^*).